ELEN E3401: Electromagnetics
Spring 2025

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Lecture #8

COLUMBIA ENGINEERING
The Fu Foundation School of Engineering and Applied Science

Wave Impedance - summary

Wave impedance: $Z(d) = \frac{\tilde{V}(d)}{\tilde{I}(d)} = Z_0\left(\frac{1+\Gamma_d}{1-\Gamma_d}\right)$ $\qquad \Gamma_d = \Gamma e^{-j2\beta d} = |\Gamma| e^{j(\theta_r - 2\beta d)}$

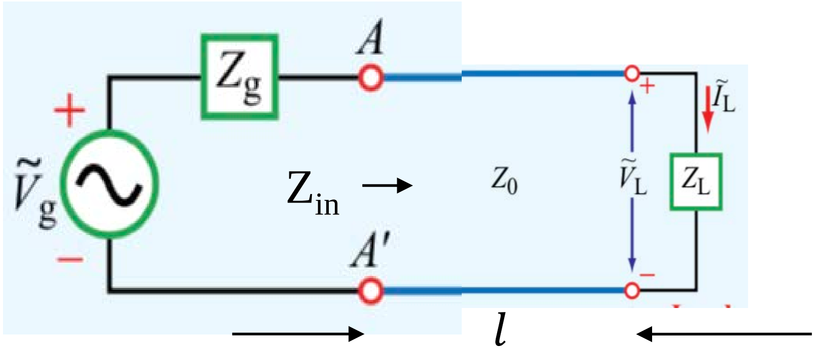
Input impedance at d = l :

$$Z_{in} = Z(l) = Z_0\left(\frac{1+\Gamma e^{-j2\beta l}}{1-\Gamma e^{-j2\beta l}}\right) = Z_0\left(\frac{z_L + j\tan(\beta l)}{1 + j z_L \tan(\beta l)}\right) \qquad z_L = \frac{Z_L}{Z_0}$$

We obtain solution:

$$V_0^{\ +} = \left(\frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}}\right)\left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right)$$

Now we can obtain full *v(d,t), i(d,t)* Solutions: given $V_g(t) \rightarrow \tilde{V}_g$, Z_g, Z_0, Z_L, length TL



obtain→ $\tilde{V}_g$, Z_{in}, Γ, β

Special cases of lossless TL

$$\tilde{V}(d) = V_0^+(e^{j\beta d} + \Gamma e^{-j\beta d})$$

$$\tilde{I}(d) = \frac{V_0^+}{Z_0} V_0^+(e^{j\beta d} - \Gamma e^{-j\beta d})$$

$$\mathrm{Z(d)} = Z_0\left(\frac{1+\Gamma_d}{1-\Gamma_d}\right)$$

$$\mathrm{Z(d)} = \frac{\tilde{V}(d)}{\tilde{I}(d)} = \frac{V_0^+(e^{j\beta d} + \Gamma e^{-j\beta d})}{V_0^+(e^{j\beta d} - \Gamma e^{-j\beta d})} Z_0$$

$$\Gamma_d = \Gamma e^{-j2\beta d} = |\Gamma| e^{j\theta_r - j2\beta d} = |\Gamma| e^{j(\theta_r - 2\beta d)}$$

Special cases of lossless TL: short circuit

$$\tilde{V}(d) = V_0^+(e^{j\beta d} + \Gamma e^{-j\beta d}) \qquad \tilde{I}(d) = \frac{V_0^+}{Z_0} V_0^+(e^{j\beta d} - \Gamma e^{-j\beta d})$$

$$\mathrm{Z(d)} = Z_0\left(\frac{1+\Gamma_d}{1-\Gamma_d}\right) \qquad \mathrm{Z(d)} = \frac{\tilde{V}(d)}{\tilde{I}(d)} = \frac{V_0^+(e^{j\beta d} + \Gamma e^{-j\beta d})}{V_0^+(e^{j\beta d} - \Gamma e^{-j\beta d})} Z_0$$

$$\Gamma_d = \Gamma e^{-j2\beta d} = |\Gamma| e^{j\theta_r - j2\beta d} = |\Gamma| e^{j(\theta_r - 2\beta d)}$$

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Short Circuit: $Z_L = 0, \Gamma = -1$:

$$\tilde{V}_{sc}(d) = V_0^+(e^{j\beta d} - e^{-j\beta d}) = 2jV_0^+ \sin(\beta d)$$

$$\tilde{I}_{sc}(d) = \frac{V_0^+}{Z_0}(e^{j\beta d} + e^{-j\beta d}) = \frac{2V_0^+}{Z_0}\cos(\beta d)$$

$$Z_{sc}(d) = \frac{\tilde{V}_{sc}(d)}{\tilde{I}_{sc}(d)} = jZ_0 \tan(\beta d)$$

Special cases of lossless TL: <u>short circuit</u>

$$\tilde{V}_{sc}(d) = V_0^+(e^{j\beta d} - e^{-j\beta d}) = 2jV_0^+\sin(\beta d)$$

$$Z_{sc}(d) = \frac{\tilde{V}_{sc}(d)}{\tilde{I}_{sc}(d)} = jZ_0\tan(\beta d)$$

$$\tilde{I}_{sc}(d) = \frac{V_0^+}{Z_0}(e^{j\beta d} + e^{-j\beta d}) = \frac{2V_0^+}{Z_0}\cos(\beta d)$$

→ The voltage $\tilde{V}_{sc}(d) = 0$ at $d = 0$ (load) → short circuit

Then as we go away from load the voltage amplitude will vary as: $\sin(\beta d)$

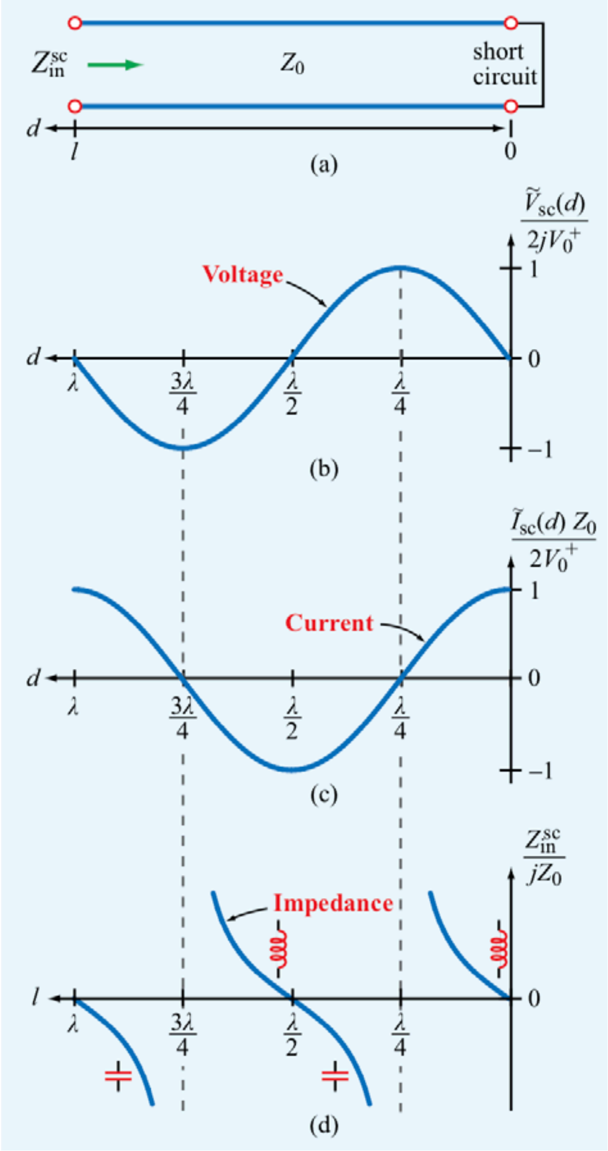
<u>Note</u>: this is the <u>amplitude</u>, not $|\tilde{V}(d)|$ which is a standing wave of $\lambda/2$

→ The current amplitude $\tilde{I}_{sc}(d)$ is maximum at load and varies as $\cos(\beta d)$

→Z_{in}^{SC}: input impedance of short-circuited line of length l:

$$Z_{in}^{SC} = \frac{\tilde{V}_{sc}(l)}{\tilde{I}_{sc}(l)} = jZ_0\tan(\beta l)$$

Special cases of lossless TL: short circuit



$$Z_L = 0, \Gamma = -1, S = \infty$$

Normalized voltage amplitude

Normalized current amplitude

Normalized input impedance

Special cases of lossless TL: short circuit

$$Z_{in}^{SC} = \frac{\tilde{V}_{sc}(l)}{\tilde{I}_{sc}(l)} = jZ_0 \tan(\beta l)$$

In general, $Z_{in} = R_{in} + jX_{in}$ ← Input reactance

Input impedance   Input resistance

For short circuit, $Z_{in} = jX_{in}$ (purely reactive)

For SC line, if $l < \lambda/4$, its impedance is equivalent to an <u>inductor</u>

If $\lambda/4 < l < \lambda/2$ → impedance looks like a <u>capacitor</u>

$$Z_{in}{}^{sc} = jZ_0 \tan(\beta l) = jZ_0 \tan\left(\frac{2\pi}{\lambda}\frac{\lambda}{8}\right) = jZ_0 \tan(\frac{\pi}{4})$$

$$l < \frac{\lambda}{4}, \tan(\beta l) > 0 \qquad Z_{in}^{SC} = jZ_0(tan) \dashrightarrow inductor$$

$$\frac{\lambda}{4} \le l < \frac{\lambda}{2}, \tan(\beta l) < 0 \qquad Z_{in}^{SC} = -jZ_0(tan) \dashrightarrow capacitor$$

Special cases of lossless TL: <u>short circuit</u>

For $\tan(\beta l) \geq 0$ → line appears inductive with equivalent inductance = L_{eq},

$$\mathrm{j}\omega L_{eq} = jZ_0 \tan(\beta l)$$

$$L_{eq} = \frac{Z_0 \tan(\beta l)}{\omega} \ (H) \qquad \text{if } \tan(\beta l) \geq 0$$

Minimum length of Z_{in}^{SC} equivalent to inductor: $l = \frac{1}{\beta} tan^{-1}(\frac{\omega L_{eq}}{Z_0})$ (m)

For $\tan(\beta l) \leq 0$ → input impedance is <u>capacitive</u>, line acts like equivalent capacitor,

$$\frac{1}{\mathrm{j}\omega C_{eq}} = jZ_0 \tan(\beta l) \qquad \text{if } \tan(\beta l) \leq 0$$

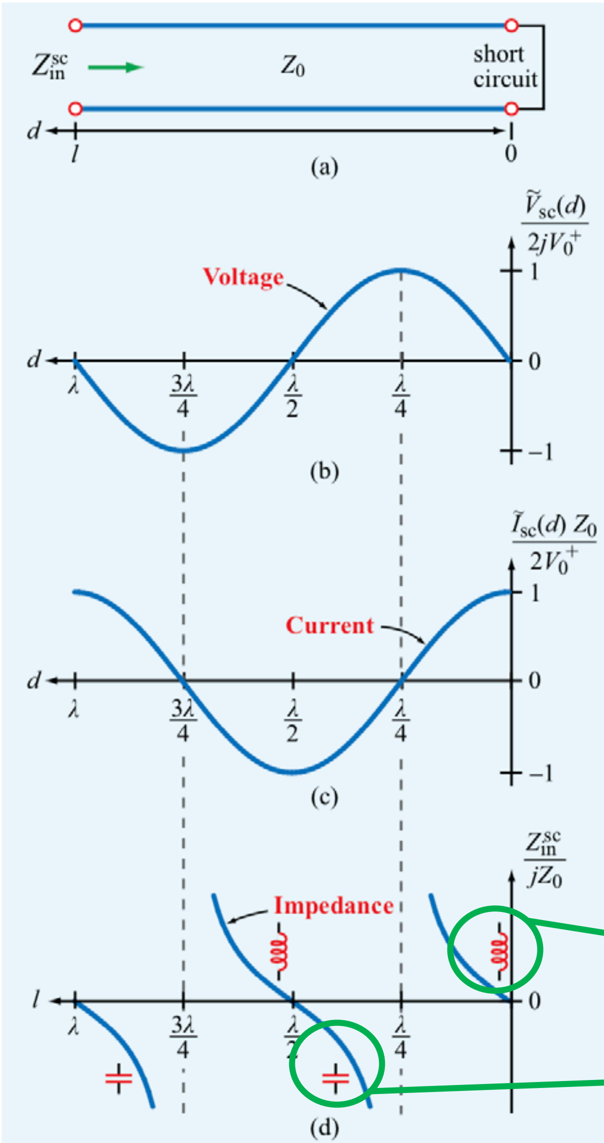
$$C_{eq} = \frac{1}{Z_0 \omega \tan(\beta l)} \ (F)$$

Shortest length of l for $\tan(\beta l) \leq 0$ is: $l = \frac{1}{\beta}[\pi - tan^{-1}(\frac{1}{\omega C_{eq} Z_0})]$ (m) $\quad \frac{\pi}{2} \leq \beta l \leq \pi$

Make <u>equivalent inductors and capacitors</u> by designing length of short- circuited line

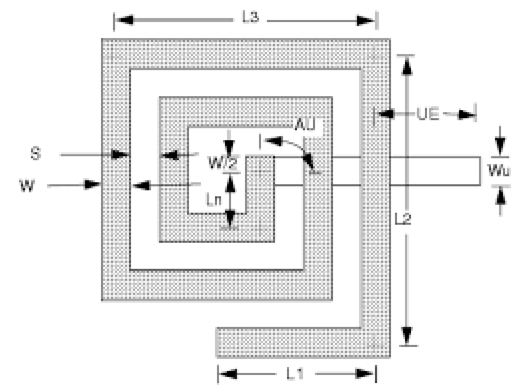
Common in high speed microwave circuits, fabricating inductors/capacitors more challenging than microstrip transmission line

Special cases of lossless TL: short circuit



$Z_L = 0, \Gamma = -1, S = \infty$

Normalized voltage amplitude



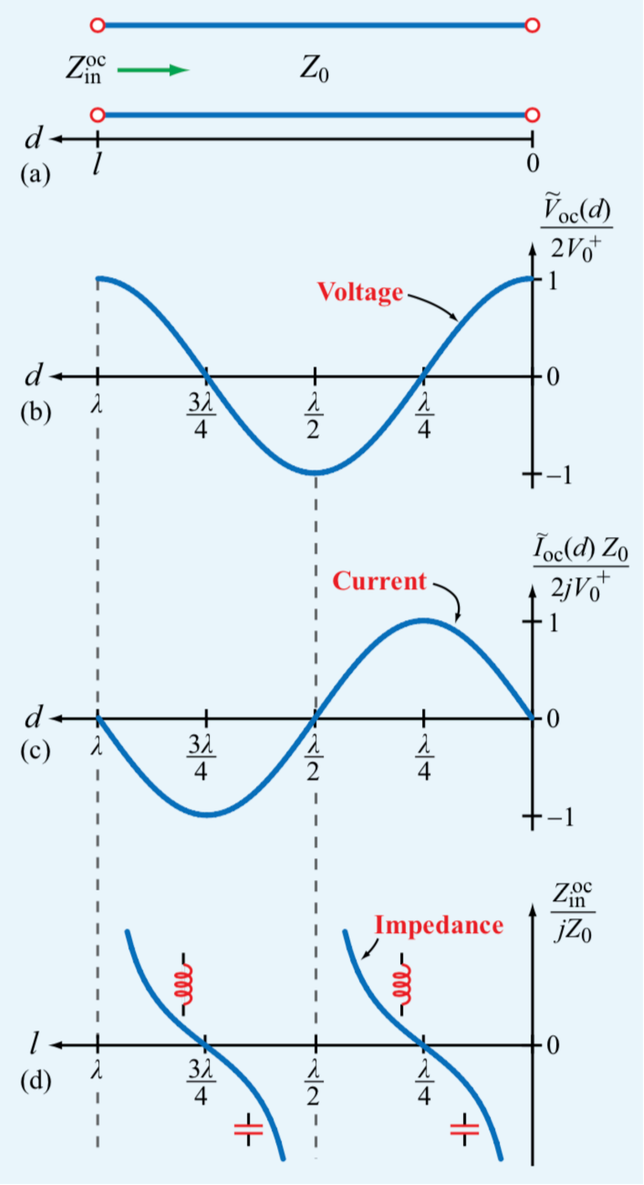
Normalized current amplitude

Normalized input impedance

$$L_{eq} = \frac{Z_0 \tan(\beta l)}{\omega} \ (H)$$

$$C_{eq} = \frac{1}{Z_0 \omega \tan(\beta l)} \ (F)$$

Special cases of lossless line: open circuit



$$Z_L = \infty, \Gamma = 1, S = \infty$$

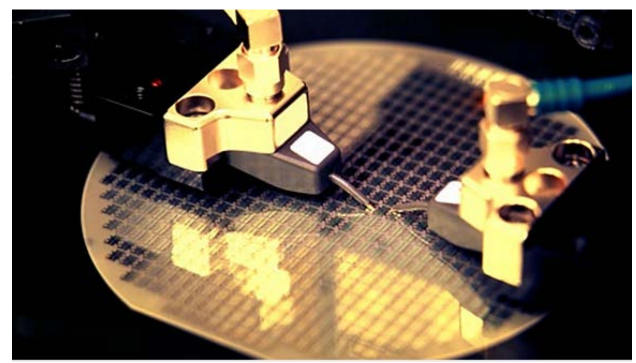
$$\tilde{V}_{oc}(d) = V_0^+(e^{j\beta d} + e^{-j\beta d}) = 2V_0^+\cos(\beta d)$$

$$\tilde{I}_{oc}(d) = \frac{V_0^+}{Z_0}(e^{j\beta d} - e^{-j\beta d}) = \frac{2jV_0^+}{Z_0}\sin(\beta d)$$

Normalized input impedance

$$Z_{in}^{oc} = \frac{\tilde{V}_{oc}(l)}{\tilde{I}_{oc}(l)} = -jZ_0\cot(\beta l)$$

Application of SC/OC Technique



<u>Network analyzer</u> – used to measure: impedance
RF – at high frequencies

Measures 1) Z_{in}^{sc} and 2) Z_{in}^{oc} → to obtain Z_0 and β

$$Z_0{}^2 = \underbrace{(jZ_0\tan(\beta l))}_{Z_{in}^{sc}}\underbrace{(-jZ_0\cot(\beta l))}_{Z_{in}^{oc}}$$

$$Z_0 = +\sqrt{Z_{in}^{sc}Z_{in}^{oc}}$$

$$\tan(\beta l) = \sqrt{\frac{-Z_{in}^{sc}}{Z_{in}^{oc}}} \qquad l \leq \lambda/2$$

Lines of length $l = n\lambda/2$

For $l = n\lambda/2$, (n = 1,2, …) $\tan(\beta l) = \tan\left(\frac{2\pi}{\lambda}\frac{n\lambda}{2}\right) = \tan(n\pi) = 0$

$$Z_{in} = Z_0\left(\frac{z_L + j tan(\beta \mathrm{l})}{1 + j z_L tan(\beta \mathrm{l})}\right) = Z_0\left(\frac{Z_L}{Z_0}\right) = Z_L$$

$Z_{in} = Z_L$ Half-wavelength line **does not** modify load impedance

Lines of length $l = \lambda/4$ (quarter-wave)

For $l = \lambda/4 + n\lambda/2$, (n = 0, 1, 2, …) $\beta l = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2}$

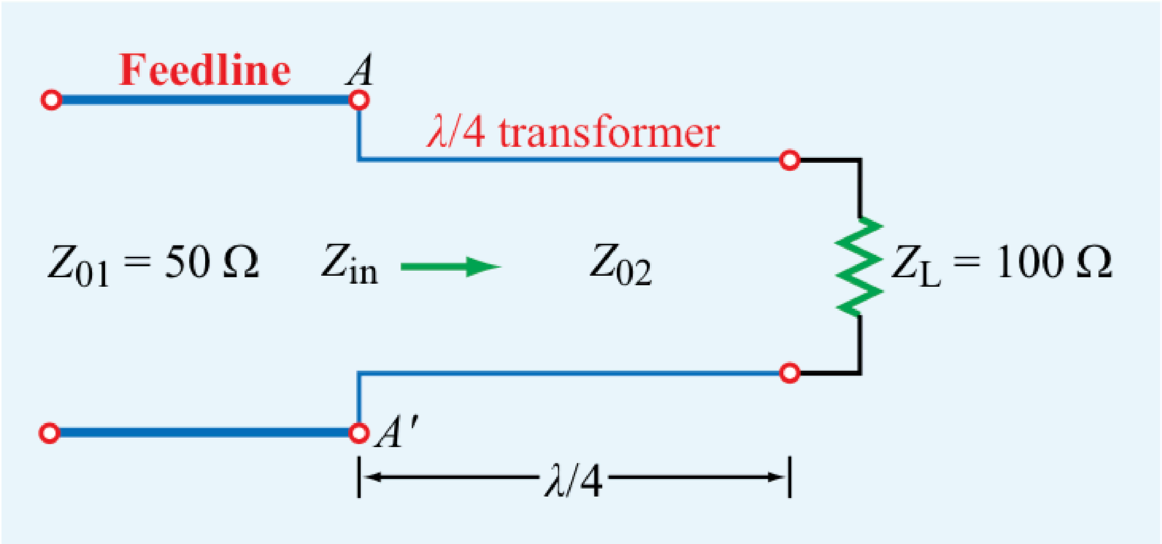
$$Z_{in} = Z_0\left(\frac{z_L\cos(\beta l) + jsin(\beta \mathrm{l})}{\cos(\beta l) + jz_L sin(\beta \mathrm{l})}\right) = Z_0\left(\frac{j}{jz_L}\right)$$

$$Z_{in} = \frac{{Z_0}^2}{Z_L} \qquad l = \lambda/4 + n\lambda/2$$

λ/4 transformer

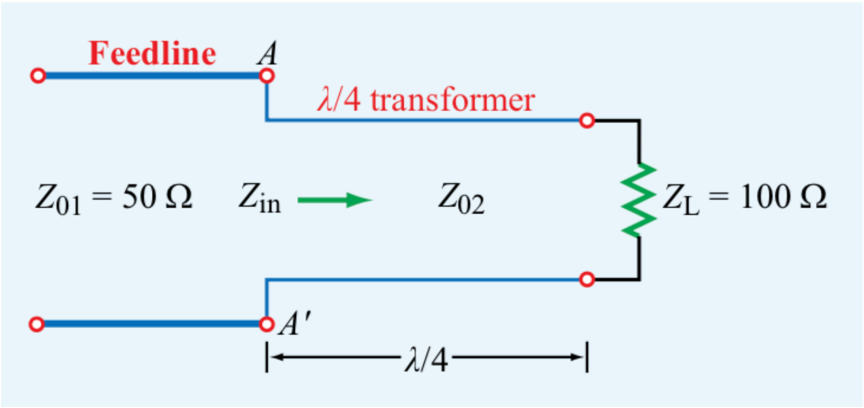
We have a 50 Ω lossless transmission line

We want to match to $Z_L = 100\ \Omega$ using a quarter wave section



Find: required impedance of λ/4 transformer

λ/4 transformer



To match the feedline, need to eliminate reflections at AA'. → $Z_{in} = 50\ \Omega = Z_{01}$

Quarter-wave TL: $Z_{in} = \dfrac{{Z_{02}}^2}{Z_L}$ $\quad Z_{02} = \sqrt{Z_{in} Z_L} = \sqrt{50 \times 100} = 70.7\ \Omega$

<u>Matched transmission: </u> $Z_L = Z_0$

- $Z_{in} = Z_0$ for all *d* on the line
- $\Gamma = 0$
- All incident power delivered to load